BLOWING UP IN THE CATEGORY OF FORMAL COMPLEX SPACES(Real Singularities and Real Algebraic Geometry)

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BLOWING UP IN THE CATEGORY OF FORMAL COMPLEX SPACES

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Introduction

This is an introductory note on blowings up of ideals on formal complex spaces. They share many properties with blowings up of ideals on complex spaces. Hence they may be known or expected. But analogy often misleads us in formal geometry. So I think it necessary to certify them before use. (I am preparing a paper on convergence of formal morphisms.)

There are two main inconveniences in formal geometry in comparison with analytic geometry. First, the underlying topological space carries little information. Second, we can not use the Weierstraß preparation theorem freely. Consequently we must replace some geometric arguments in analytic geometry by more algebraic ones and sometimes we are forced to treat local rings through their "full completions". Nevertheless our arguments are also effective in analytic geometry, because complex spaces form a subcategory of the category of formal complex spaces. (So this note will serve as a consistent introducton to the theory of blowing up in complex analytic geometry also.) The firm foundation was already laid by

Bingener. When we treat dimension (§4 and §5), we rely upon many results from algebra: theories of Pfaffian forms on local algebras, formally equidimensional rings (=quasi-unmixed rings), excellent rings etc.

Let us list some basic definitions and notations about formal complex spaces. Let $\mathbb{X}\equiv(|\mathbb{X}|, \mathcal{O}_{\mathbb{X}})$ be a local $\mathbb{C}$-ringed space and $S\subset|\mathbb{X}|$ a subset. (See [GR₂], [H], [Mo] for ringed spaces.) Let $I_S\subset\mathcal{O}_{\mathbb{X}}$ denote the ideal sheaf defined by the following:

$$I_S(U)\equiv\{f\in\mathcal{O}_{\mathbb{X}}(U):\ f_\xi\in m_\xi \text{ for any } \xi\in S\cap U\}$$

(m_ξ: the maximal ideal of $\mathcal{O}_{X,\xi}$). Then $\mathbb{X}$ is a *formal complex space* if the following conditions are satisfied (Krasnov [K]).

(1) For any $n\in\mathbb{N}$, $\mathbb{X}_n\equiv(|\mathbb{X}|, \mathcal{O}_{\mathbb{X}}/I_{|\mathbb{X}|}{}^n)$ is a complex space;

(2) The canonical homomorphism $\mathcal{O}_{\mathbb{X}}\longrightarrow\varprojlim\mathcal{O}_{\mathbb{X}}/I_{|\mathbb{X}|}{}^n$ is an isomorphism.

(By the embedding theorem [B₂], (1.7) of formal analytic spaces, we can paraphrase this in a constructive way (see [AT], p.5, p.6), which may be familiar to analytic geometers.) A *morphism* between formal complex spaces is a morphism in the category of local $\mathbb{C}$-ringed spaces. We abbreviate complex space (resp. formal complex space) to CS (resp. FCS). The category of CSs (resp. FCSs) is expressed as $\mathbb{CS}$ (resp. $\mathbb{FCS}$). Thus $\Phi\in\mathbb{CS}:X\longrightarrow Y$ (resp. $\Phi\in\mathbb{FCS}:\mathbb{X}\longrightarrow\mathbb{Y}$) implies a morphism between CSs (resp. FCSs).

The structure sheaf $\mathcal{O}_{\mathbb{X}}$ of $\mathbb{X}\in\mathbb{FCS}$ is known to be coherent over itself ([B₂], (1.4)). Hence many argument in $\mathbb{CS}$ can be transferred to $\mathbb{FCS}$. By *subspace* of an FCS (or a CS) we mean a closed subspace defined by a coherent ideal sheaf unless it is specified otherwise. An ideal sheaf $I \subset \mathcal{O}_{\mathbb{X}}$ is an *ideal of definition* if it is coherent and, for any $\xi \in |\mathbb{X}|$, there exist its neighborhood U and $k \in \mathbb{N}$ such that $I_{|\mathbb{X}|}{}^k \subset I \subset I_{|\mathbb{X}|}$ on U. $I_{|\mathbb{X}|}$ is an ideal of definition. An ideal sheaf on an FCS $\mathbb{X}$ is *analytic* if it is coherent and includes an ideal of definition of $\mathbb{X}$. Then a subset $S \subset |\mathbb{X}|$ is analytic in the complex space $\mathbb{X}_n$ (of (1) above) if and only if I_S is analytic. A morphism $\Phi \in \mathbb{FCS}: \mathbb{X} \longrightarrow \mathbb{Y}$ is *adic* if an (every) ideal of definition of $\mathbb{Y}$ generates an ideal of definition of $\mathbb{X}$. Hence Φ is adic if and only if the inverse image ideal sheaves of all (some) analytic ideal sheaves are analytic. The inclusion morphism of a subspace (or an open subspace) is adic.

Let $S \subset |\mathbb{X}|$ be an analytic subset of an FCS. Then $\hat{\mathbb{X}}_{|S} \equiv (S, \hat{\mathcal{O}}_{\mathbb{X}|S})$ with $\hat{\mathcal{O}}_{\mathbb{X}|S} \equiv \varprojlim \mathcal{O}_{\mathbb{X}} / I_S{}^k$ is an FCS. We call $\hat{\mathbb{X}}_{|S}$ the (*formal*) *completion of* $\mathbb{X}$ *along* S. Let $S \subset |\mathbb{X}|$ and $T \subset |\mathbb{Y}|$ be analytic subsets of FCSs and suppose that $\Phi \in \mathbb{FCS}: \mathbb{X} \longrightarrow \mathbb{Y}$ satisfy $|\Phi|(S) \subset T$. Then Φ induces a morphism $\hat{\Phi} \equiv \hat{\Phi}_{S;T} \in \mathbb{FCS}: \hat{\mathbb{X}}_{|S} \longrightarrow \hat{\mathbb{Y}}_{|T}$, which we call the *completion* of Φ. Completion of an FCS and that of a morphism in $\mathbb{FCS}$ are compatible with restriction to open subsets. If $\Psi \in \mathbb{FCS}: \mathbb{Y} \longrightarrow \mathbb{Z}$ s.t. $|\Psi|(T) \subset R$, we have $\hat{\Psi}_{T;R} \cdot \hat{\Phi}_{S;T} = (\Psi \cdot \Phi)\hat{}_{S;R}$.

We call a local $\mathbb{C}$-algebra a *formal analytic algebra* if it is isomorphic to a stalk algebra $\mathcal{O}_{\mathbb{X},\xi}$ at a point of an FCS. A germ at a point of an FCS (or CS) is called *integral* if its local ring is integral i.e. reduced and irreducible. For $\xi \in S$, we have the homomorphisms

$$\mathcal{O}_{\mathbb{X},\xi} \xrightarrow{\widehat{\iota}_{S:|X|}} \widehat{\mathcal{O}}_{\mathbb{X}|S,\xi} \xrightarrow{\widehat{\iota}_{\xi:S}} \widehat{\mathcal{O}}_{\mathbb{X},\xi} \equiv \mathcal{O}^{\wedge}_{\mathbb{X},\xi}$$

corresponding to the completions $\hat{I}_{S:|X|}$ and $\hat{I}_{\xi:S}$ of the identity I. These homomorphisms are faithfully flat (cf. [Mat],§7,§8). Hence they are injective and can be considered as inclusions: $\mathcal{O}_{X,\xi} \subset \widehat{\mathcal{O}}_{X|S,\xi} \subset \widehat{\mathcal{O}}_{X,\xi}$. Since $\widehat{\mathbb{X}}_{||X|}$ is canonically isomorphic to $\mathbb{X}$, $\mathbb{CS}$ is a full subcategory of $\mathbb{FCS}$.

Let $\Phi_\xi : X_\xi \longrightarrow Y_\eta$ $(\eta = \Phi(\xi))$ denote the germ of morphism $\Phi \in \mathbb{FCS}$ at ξ. Then the canonical homomorphism ("composition with Φ") between the local rings by the corresponding small Greek letter as $\phi_\xi : \mathcal{O}_{\mathbb{Y},\eta} \longrightarrow \mathcal{O}_{\mathbb{X},\xi}$. If F is a sheaf on $\mathbb{X} \in \mathbb{FCS}$ (with some algebraic structure), F(A) denotes the set of sections of F over a subset $A \in |\mathbb{X}|$ (with the corresponding structure). If $|\mathbb{X}|$ is Hausdorff, A has a foundamental system of paracompact neighborhoods of A and $F(A) = \varinjlim F(U)$ where U runs over such a system (cf. [Go], II, 3.3).

This is a rearrangement of the basic part of my speechs at symposiums held in Nagoya Univ. (Dec. 1991), RIMS Kyoto Univ. (June 1992) and Nagano Nat. College of Tech. (Aug. 1992).

1. Curve selection lemma

We call an element f of a local ring A *active* if f mod N_A is a nonzerodivisor of $A_{red} \equiv A/N_A$, where $N_A \equiv \sqrt{0}$ denotes the nilradical of A. A product of active elements is active. A ring of finite Krull dimension is *equidimensional* if dim A/p=dim A for any minimal prime ideal p of A. Let I be an ideal of an equidimensional local ring A. Then I contains an active element if and only if dim A/I<dim A. (Reduce to the case A is integral.) We call a subspace $\mathbb{Y}$ of an FCS $\mathbb{X}$ *thin at* ξ (or $\mathbb{Y}_\xi$ *is thin*) if the ideal $I_{\mathbb{Y},\xi} \subset O_{\mathbb{X},\xi}$ of $\mathbb{Y}_\xi$ contains an active element. The points where $\mathbb{Y}$ is not thin form an analytic subset. (We may assume that $\mathbb{X}$ is reduced. Apply the coherence [GR_2], (A.4.5) of the annihilator ideal sheaf of the ideal sheaf that defines $\mathbb{Y}$.) Note that the underlying topological space of a thin subspace of an FCS is not always topologically thin: The reduced one point $(\{0\},\mathbb{C})$ is a thin subspace of $\hat{\mathbb{C}}_{|0} \equiv (\{0\},\mathbb{C}[t])$ but their share the same underlying topological space.

1.1. Lemma. Let A and B be equidimensional local rings and $\phi : A \longrightarrow B$ a finite injective homomorphism. Then, $f \in B$ is active if and only if $\phi^{-1}(fB)$ includes an active element.

Proof. Since ϕ is finite and injective, dim A=dim B by Cohen-Seidenberg theorem. The induced homomorphism $A/\phi^{-1}(fB) \longrightarrow B/fB$ is also finite and injective. Hence

$\dim A/\phi^{-1}(fB) = \dim B/fB$. Thus $\dim B/fB < \dim B$ if and only if $\dim A/\phi^{-1}(fB) < \dim A$. Since A and B are equidimensional, this completes the proof. ∎

When we treat an FCS X, we have no point outside the *core* $|X|$. Hence we need some antennae to feel outside. For this purpose we choose formal curves on $\mathbb{X}$. A *formal curve on* $\mathbb{X}$ (expressed as $\mathbb{K} \subset \mathbb{X}$) is simply a morphism $\mathbb{K} \in \mathbb{FCS}: \widehat{\mathbb{C}}_{|\{0\}} \equiv (\{0\}, \mathbb{C}[\![t]\!]) \longrightarrow \mathbb{X}$. This is determined by the *initial point* $\xi \in S$ and the induced $\mathbb{C}$-homomorphism $\kappa : \mathcal{O}_{X,\xi} \longrightarrow \mathbb{C}[\![t]\!]$. Note that we do not assume that $\mathbb{K}$ is adic (in contrast to many papers that deal with formal modifications). Hence the image of $\mathbb{K}$ does not always bear a structure of a subspace of $\mathbb{X}$.

<u>1.2.</u> <u>Lemma.</u> (Curve selection lemma.) Let $\mathbb{Y}$ be a subspace of an FCS $\mathbb{X}$ which is thin at $\xi \in |\mathbb{X}|$, there exists a formal curve $\mathbb{K} \subset \mathbb{X}$ with initial point ξ such that $\mathbb{K} \not\subset \mathbb{Y}$.

Proof. We may assume that $\mathbb{X}$ is irreducible at ξ. Let $\widehat{\mathbb{X}} \equiv \mathbb{X}_{|\xi} \equiv (\{\xi\}, \widehat{\mathcal{O}}_{X,\xi})$ be the full completion of $\mathbb{X}$ at ξ. Every formal curve on $\widehat{\mathbb{X}}$ naturally yields one on $\mathbb{X}$. Since $\widehat{\mathcal{O}}_{X,\xi}$ is flat over $\mathcal{O}_{X,\xi}$, the image of an active element is also active. This means that the inverse image of $\mathbb{Y}$ in $\widehat{\mathbb{X}}$ is also thin. Then we have only to work with $\widehat{\mathbb{X}}$ instead of $\mathbb{X}$. Thus we may assume that $A \equiv \mathcal{O}_{X,\xi}$ is integral and fully formal i.e. A is an integral residue class algebra of $F_n \equiv \mathbb{C}[x_1, \dots, x_n]$. It is well-known that there exists a finite injective homomorphism $\lambda : F_r \longrightarrow A$ ($r = \dim A \leqq n$). Let $I_{\mathbb{Y}} \subset A$ be the ideal of $\mathbb{Y}$. By (1.1), $\lambda^{-1}(I_{\mathbb{Y}})$ includes an active element and defines a thin subspace $\mathbb{H} \subset \hat{\mathbb{C}}^r{}_{|0}$. There exists a formal curve $\mathbb{\Gamma} \subset \hat{\mathbb{C}}^r{}_{|\{0\}}$ which is included neither in $\mathbb{H}$ nor in the discriminant locus of the projection $\mathbb{\Lambda}$ corresponding to λ (an easy case of the lemma). Replacing the parameter of $\mathbb{\Gamma}$ by its radical $\sqrt[p]{t}$ if necessary, we have a lifting $\mathbb{K} \subset \mathbb{X}$ of $\mathbb{\Gamma}$ (cf. [To], (1.6)). $\mathbb{\Gamma} \not\subset \mathbb{H}$ implies $\mathbb{K} \not\subset \mathbb{Y}$. ∎

This lemma yields a variant of Rückert Nullstellensatz in analytic geometry (cf. [Mo], (I.5.3.1)). (We do not use (1.3) later.)

<u>1.3.</u> <u>**Theorem.**</u> (Nullstellensatz.) Let $\mathbb{X}$ be an FCS $C \subset |\mathbb{X}|$ a compact set and $\mathbb{Y} \subset \mathbb{X}$ a subspace thin at all point of C. Then $f \in \mathcal{O}_{\mathbb{X}}(C)$ is nilpotent, if and only if $\kappa(f) = 0$ for any formal curve $\mathbb{K} \subset \mathbb{X}$ with the initial point in C such that $\mathbb{K} \not\subset \mathbb{Y}$.

Proof. We have only to prove the case $C \equiv \{\xi\}$ (one point). Since nilpotent elements in $\mathbb{C}[t]$ are zero, the necessity is obvious. Suppose that f is not nilpotent. Then f is active on some irreducible component $\mathbb{Z}_\xi$ of the reduction of $\mathbb{X}_\xi$. Let $\mathbb{Z}$ be a representative of $\mathbb{Z}_\xi$ in a small neighborhood U. We may assume that f has a representative f~ in U. Then the subspace $\mathbb{W} \subset U$

defined by $f^{\sim}$ is thin at ξ. $\mathbb{Y} \cup \mathbb{W}$ is also thin in $\mathbb{Z}$ and there exists a formal curve $\mathbb{K} \subset \mathbb{Z} \subset \mathbb{X}$ such that $\mathbb{K} \not\subset \mathbb{Y} \cup \mathbb{W}$ by (1.2). Hence $\kappa(f) \neq 0$ and $\mathbb{K} \not\subset \mathbb{Y}$. This proves the sufficiency. ∎

2. Minimality condition

The exceptional space of a blowing up of a CS satisfies the minimality condition as observed by Hironaka ([H_1], p. 127, [H_2], (2.7)). Minimality of $Y \subset X \in \mathbb{CS}$ is the following.

(*) If Z is a subspace of X that coincides with X outside $|Y|$, then Z=X.

This is a neat expression but in formal case this does not suit our purpose. Indeed, if $|\mathbb{X}| = |\mathbb{Y}|$ (*) means nothing. Thus we adopt the following paraphrases (using the idea of gap sheaves of Thimm, cf. [T], [ST]).

2.1. Proposition. Let Y be a subspace of $X \in \mathbb{CS}$ defined by coherent ideal sheaf I. The condition (*) is equivalent to each of the following:

(**) If $J \subset \mathcal{O}_X$ is a coherent ideal sheaf such that $I^k J = 0$ for some $k \in \mathbb{N}$ locally, then $J = 0$.

(***) If $J \subset \mathcal{O}_X$ is a coherent ideal sheaf such that $I^k \cap J = 0$ for some $k \in \mathbb{N}$ locally, then $J = 0$.

Proof. Suppose (**) and that Z coincides with X outside $|Y|$. If

$J \subset O_X$ denotes the ideal sheaf of Z, it is obvious that $J=0$ on $|Y|^c$. Then, if we apply the version of the Rückert Nullstellensatz by Grauert-Remmert ([GR_2], (3.2.2)) to local sections which generate I, we see that $I^k J=0$ for sufficiently large $k \in \mathbb{N}$ locally. Hence $J=0$ by (**) and (*) holds. (*) trivially implies (**).

Since the ring of sections $O_X(S)$ is Noetherian for any semianalytic Stein compact set S by the theorem of Frisch, Grothendieck and Siu, we have the Artin-Rees equality $I^{k+p} \cap J = I^k(I^p \cap J)$ for some $p \in \mathbb{N}$ for any $k \in \mathbb{N}$ locally (cf. [Mat], (8.5)). Hence $I^{k+p} J \subset I^{k+p} \cap J \subset I^k J$ ($k \in \mathbb{N}$) holds locally and (**) and (***) are equivalent. ∎

Bingener has proved that $O_X(S)$ is Noetherian for any semianalytic Stein compact set in the formal-analytic case also ([B_2], (1.4)). Thus we have the following in just the same way as the last part of Proof above.

2.2. Proposition. (**) and (***) are equivalent for FCSs $\mathbb{Y} \subset \mathbb{X}$ also.

Let $\mathbb{Y}$ be a subspace of $\mathbb{X} \in \mathbb{FCS}$ defined by a coherent ideal sheaf I. We say that $(\mathbb{X}, \mathbb{Y})$ satisfies the *minimality condition* if it satisfies one of equivalent condition (**) and (***).

2.3. Theorem. (cf. [H_2], (2.7.4)) Let $\mathbb{Y} \subset \mathbb{X} \in \mathbb{FCS}$ be a subspace defined by a coherent ideal sheaf I. Then we have the following:

(i) $J \equiv \bigcup_{k \geq 1} \mathrm{Ann}\, I^k \subset O_{\mathbb{X}}$ is locally stationary and coherent;

(ii) J is the largest ideal sheaf that satisfies $I^k J = 0$ for some $k \in \mathbb{N}$ locally;

(iii) J is the largest ideal sheaf that satisfies $I^k \cap J = 0$ for some $k \in \mathbb{N}$ locally;

(iv) J defines the subspace $\mathbb{Z} \subset \mathbb{X}$ such that $(\mathbb{Z}, \mathbb{Y} \cap \mathbb{Z})$ satisfies the minimality condition. (i.e. $\mathbb{Z}$ is the *formal-analytic closure* of $\mathbb{X} | |\mathbb{Y}|^c$ in $\mathbb{X}$.)

Proof. (i) *Ann* I^k is a coherent ideal sheaf (cf. [GR_2], (A.4.5)). The assertions follow from the Noetherian property ([B_2], (1.5)).

(ii) Then $J = \mathit{Ann}\, I^k$ for some k and $I^k J = 0$ locally. The maximality follows from the definition of J.

(iii) Obvious from (ii) and the local inclusion

$$I^{k+p} J \subset I^{k+p} \cap J = I^k (I^p \cap J) \subset I^k J.$$

(iv) We may work at a point ξ. Let $\iota_\xi : O_{\mathbb{X},\xi} \longrightarrow O_{\mathbb{Z},\xi}$ denote the canonical epimorphism. $\iota_\xi(I_\xi) \subset O_{\mathbb{Z},\xi}$ is the ideal of $(\mathbb{Y} \cap \mathbb{Z})_\xi$. Suppose that $K_\xi \subset O_{\mathbb{Z},\xi}$ is an ideal such that $\iota_\xi(I_\xi)^l K_\xi = 0$ for some $l \in \mathbb{N}$. Then $I_\xi{}^l \iota_\xi^{-1}(K_\xi) \subset J_\xi$. Since $J_\xi = \mathit{Ann}\, I_\xi{}^k$, $I_\xi{}^{k+l} \iota_\xi^{-1}(K_\xi)) = 0$. Then, by the definition of J_ξ, we have $\iota_\xi^{-1}(K_\xi) \subset J_\xi$ i.e. $K_\xi = 0$. This verifies the minimality condition. ∎

<u>2.4.</u> <u>Corollary.</u> Let $\mathbb{Y} \subset \mathbb{X} \in \mathbb{FCS}$ be a subspace defined by a coherent ideal sheaf I. $(\mathbb{X}, \mathbb{Y})$ satisfies the minimality condition if and only if $Ann\ I^k = 0$ for all $k \in \mathbb{N}$.

An ideal sheaf I on an $\mathbb{FCS}$ $\mathbb{X}$ is called *invertible* if I is locally isomorphic to $O_{\mathbb{X}}$ as an $O_{\mathbb{X}}$-module i.e. I is generated by a nonzerodivisor locally. Then the following is obvious.

<u>2.5.</u> <u>Corollary.</u> Let $I \neq 0$ be a locally principal ideal sheaf on an FCS $\mathbb{X}$ and $\mathbb{Y}$ its subspace defined by I. Then the following conditions are equivalent:

(i) $(\mathbb{X}, \mathbb{Y})$ satisfies the minimality condition.

(ii) I is invertible i.e. $\mathbb{Y}$ is a *hypersurface* of $\mathbb{X}$.

3. Formal blowing up

Let $\mathbb{X}$ be a Stein formal complex space (i.e. $\exists$ or $\forall\, \mathbb{X}_n$ is Stein). Bingener has constructed a formal complex space Z^{an} corresponding to every $\mathcal{O}_{\mathbb{X}}(|\mathbb{X}|)$-scheme Z locally of finite presentation ([B_2]) such that Z^{an} satisfies nice functorial properties. This enables us (1) to construct the (formal) analytic projective spectrum Projan G for a sheaf G of graded algebras over an FCS which is locally generated by sections of degree 1 and of finite presentation and then (2) to define blowing up in $\mathbb{FCS}$. These constructions are quite similar to $\mathbb{CS}$ case (cf. [Ho], [B_1], §5, [Mo], III, §1) as follows.

Let $I \subset \mathcal{O}_{\mathbb{X}}$ be a coherent ideal sheaf on an FCS $\mathbb{X}$. Let us consider the graded $\mathcal{O}_{\mathbb{X}}$-algebra $G_I \equiv \bigoplus_{k\geq 0} I^k$. For any $\xi \in |\mathbb{X}|$, take its small neighborhood U and sections $g_0, \ldots, g_p \in I(\mathrm{U})$ which generate $I|\mathrm{U}$. If we substitute $\mathrm{T}_0, \ldots, \mathrm{T}_p$ by germs of $g_0, \ldots, g_p$ respectively, we have a degree-preserving $\mathcal{O}_{\mathbb{X}}|\mathrm{U}$-epimorphism

$$\iota_{\mathrm{U}} : \mathcal{O}_{\mathbb{X}}|\mathrm{U}[\mathrm{T}_0, \ldots, \mathrm{T}_p] \longrightarrow G_I|\mathrm{U} \equiv \bigoplus_{k\geq 0} (I|\mathrm{U})^k .$$

Here $\mathcal{O}_{\mathbb{X}}[\mathrm{T}_0, \ldots, \mathrm{T}_p]$ is graded by $\deg \mathrm{T}_i \equiv 1$ $(i=0, \ldots, p)$. Obviously the kernel $K|\mathrm{U}$ is homogeneous. It is just the *ideal sheaf of homogeneous polynomial relations among* $g_0, \ldots, g_p$. Now that we have defined the sheaf $K|\mathrm{U}$, we can transfer the coefficient of the algebras from the sheaf to the ring of their local sections as follows. If U is sufficiently small, $K|\mathrm{U}$ is known to be generated by a finite number of homogeneous elements of $K(\mathrm{U})$ by Noetherian property of ideals ([B_2], (1.5), cf.

[Mo], (1.2.7) for how to apply). Thus we have a local exact sequence:

$$0 \longrightarrow K(\mathrm{U}) \longrightarrow \mathcal{O}_{\mathbb{X}}(\mathrm{U})[\mathrm{T}_0, \ldots, \mathrm{T}_p] \xrightarrow{\iota(\mathrm{U})} G_I(\mathrm{U}) \equiv \bigoplus_{k \geq 0} I(\mathrm{U})^k \longrightarrow 0$$

such that $\iota(\mathrm{U})$ is degree preserving, $K(\mathrm{U})$ is finitely generated and $G_I(\mathrm{U})_\xi \equiv \bigoplus_{k \geq 0} I_\xi{}^k$ for any $\xi \in \mathrm{U}$ (thus G_I satisfies the condition described in (1) above). Let $\mathrm{F}_1, \ldots, \mathrm{F}_q \in K(\mathrm{U})$ be a system of homogeneous generators of $K|\mathrm{U}$ and $\mathrm{C}_I(\mathbb{X}|\mathrm{U}) \equiv \mathrm{Specan}\ G_I|\mathrm{U} \subset \mathbb{X}|\mathrm{U} \times \mathbb{C}^{p+1}$ the subspace defined by the equations $\mathrm{F}_1 = \cdots = \mathrm{F}_q = 0$. $\mathrm{C}_I(\mathbb{X}|\mathrm{U})$ is a family of cones parametrized by U. Its projectivization is denoted by $\mathrm{B}_I(\mathbb{X}|\mathrm{U}) \equiv \mathrm{Projan}\ G_I|\mathrm{U} \subset \mathbb{X}|\mathrm{U} \times \mathbb{P}^p$. The natural morphism $\mathcal{O}_{\mathbb{X}}(\mathrm{U}) \longrightarrow \mathcal{O}_{\mathbb{X}}(\mathrm{U})[\mathrm{T}_0, \ldots, \mathrm{T}_p]$ induces $\Pi^0|\mathrm{U} \in \mathbb{FCS}:\mathrm{C}_I(\mathbb{X}|\mathrm{U}) \longrightarrow \mathbb{X}|\mathrm{U}$ and $\Pi|\mathrm{U} \in \mathbb{FCS}:\mathrm{B}_I(\mathbb{X}|\mathrm{U}) \longrightarrow \mathbb{X}|\mathrm{U}$. $\mathrm{B}_I(\mathbb{X}|\mathrm{U})$ and $\Pi|\mathrm{U}$ can be glued to an FCS $\mathrm{B}_I(\mathbb{X}) \equiv \mathrm{Projan}\ G_I$ and a global morphism $\Pi \in \mathbb{FCS}:\mathrm{B}_I(\mathbb{X}) \longrightarrow \mathbb{X}$. We call the FCS $\mathrm{B}_I(\mathbb{X})$ or the morphism Π the *blowing up with center I*. The subspace defined by I is also called the *center* of Π. If $\mathbb{Y} \subset \mathbb{X}$ is an open subspace, $\mathrm{B}_I(\mathbb{Y})$ is isomorphic to an open subspace of $\mathrm{B}_I(\mathbb{X})$ and their Π are compatible. If the underlying topological spaces of FCSs are Hausdorff, we can easily show that $\mathrm{B}_I(\mathbb{X})$ is also so. We use the notations above repeatedly.

Let $\Phi: \mathrm{X} \longrightarrow \mathrm{Y}$ be a morphism between ringed spaces. For an ideal sheaf $J \subset \mathcal{O}_{\mathrm{Y}}$, let $\Phi^{-1}J$ denotes the ideal sheaf generated by the pullbacks of elements of J by Φ. (To be precise, remember that a morphism Φ consists of continuous map

$|\Phi|:|X|\longrightarrow|Y|$ between underlying topological spaces and a homomorphism $\Phi^{\natural}:\mathcal{O}_Y\longrightarrow|\Phi|_*\mathcal{O}_X$ between sheaves of rings on $|Y|$, where $|\Phi|_*\mathcal{O}_X$ denotes the direct image sheaf. $\Phi^{\natural}$ canonically induces a homomorphism ${}^{\natural}\Phi:|\Phi|^{-1}\mathcal{O}_Y\longrightarrow\mathcal{O}_X$, where $|\Phi|^{-1}\mathcal{O}_Y\equiv X\times_Y\mathcal{O}_Y$ denotes the topological inverse image sheaf. If $J\subset\mathcal{O}_Y$ is an ideal sheaf, $\Phi^{-1}J$ denotes the ideal sheaf generated by the image ${}^{\natural}\Phi(|\Phi|^{-1}J)\subset\mathcal{O}_X$.) We call $\Phi^{-1}J$ the *inverse image ideal sheaf of* J. Indeed, if J is the ideal sheaf that defines a closed subspace $Z\subset Y$, $\Phi^{-1}J$ is the right object by which we should define the *inverse image (space)* $\Phi^{-1}Z\subset X$.

Φ is *proper* if it is adic and the map $|\Phi|$ between underlying topological spaces is proper.

<u>3.1. Theorem.</u> Let $\mathbb{\Pi}\in\mathbb{FCS}:\mathbb{X}'\longrightarrow\mathbb{X}$ be a blowing up with center I. Then we have the following.

(i) $\mathbb{\Pi}$ is proper;

(ii) $J\equiv\mathbb{\Pi}^{-1}I$ is invertible;

(iii) If $\mathbb{E}$ denotes the *exceptional space* (=the subspace defined by J), then $(\mathbb{X}',\mathbb{E})$ satisfies the minimality condition.

(iv) If I is invertible on an open subset $V\subset|\mathbb{X}|$, $\mathbb{\Pi}$ is an isomorphism on $|\mathbb{\Pi}|^{-1}(V)$.

Proof. (i) Obvious from the construction. (The canonical projection in the Specan construction ([B_2], §4) is always adic.)

(ii) Let us put $g_i\equiv\iota_U(T_i)\in I(U)$ and $G_i\equiv\pi(g_i)\in\mathcal{O}_{\mathbb{X}'}(U')$.

Obviously G_i are generators of J. If we put $W^0 \equiv \mathrm{Specan}\, G_I(U) \cap \{T_j \neq 0\}$, we have $G_i = G_j T_i / T_j$ $(i \neq j)$ on W^0. Then J is generated by G_j on the image $W \subset \mathrm{Projan}\, G_I(U)$ of W^0. Suppose that G_j is a zerodivisor at $\eta \in W$. Then $G_j{}^0 \equiv \pi^0(g_j)$ is also so at a point $\eta^0 \in W^0 \setminus (\mathbb{X}|U \times \{0\})$ on the generatrix corresponding to η and there exists a homogeneous $H \in O_{\mathbb{X},\xi}[T_0, \ldots, T_p] \setminus (K|U)_\xi$ such that $Hg_j \in (K|U)_\xi$. Hence we have $HT_j \equiv 0 \bmod (K|U)_\xi$ and T_j is a zerodivisor in the local ring at η^0. This contradicts the fact that T_j has nonzero evaluation at η^0. This proves that G_j is not a zerodivisor and J is invertible.

(iii) Immediate from (2.5).

(iv) Shrinking V, we may assume that $I|V$ is generated by $g \in I(V)$ and g_ξ is not a zerodivisor for any $\xi \in V$. Then the degree-preserving epimorphism $\iota_V : (O_{\mathbb{X}}|V)[T_0] \longrightarrow G_I|V$ with $\iota_V(T_0) = g \in I(V)$ is an isomorphism. This proves that $\mathbb{I}$ is an isomorphism on $|\mathbb{I}|^{-1}(V)$. ∎

Blowing up in $\mathbb{FCS}$ can be characterized by universality by the following (as in $\mathbb{CS}$ or algebraic case, cf. [H_2], [HR] for the $\mathbb{CS}$ case).

<u>3.2.</u> <u>Theorem.</u> (Universality of blowing up.) Let $I \in O_{\mathbb{X}}$ be a coherent ideal sheaf on an FCS. If $\Phi \in \mathbb{FCS}: \mathbb{Y} \longrightarrow \mathbb{X}$ is a morphism such that $\Phi^{-1} I$ is invertible, there exists a unique lifting

$\mathbb{\Psi}: \mathbb{Y} \longrightarrow B_I(\mathbb{X})$ of $\mathbb{\Phi}$.

Proof. Since the problem is local on $\mathbb{X}$ by the uniqueness, we may work on such U as taken in the construction above and assume that $I|U$ is generated by $g_0, \ldots, g_p \in \mathcal{O}_X(U)$. Since $\mathbb{\Phi}^{-1}I$ is invertible, we may assume that, for any $\eta \in |\mathbb{\Phi}|^{-1}(U)$, there exists its neighborhood $W \subset |\mathbb{\Phi}|^{-1}(U)$ such that $\mathbb{\Phi}^{-1}I$ is generated by $\phi(g_0)$ and $\phi(g_0)$ is not a zerodivisor on W. Then there exist $h_i \in \mathcal{O}_Y(W)$ such that $\phi(g_i) = \phi(g_0)h_i$. Substituting $T_0, \ldots, T_p$ by $h_0 (\equiv 1), \ldots, h_p$ respectively, we have a lifting $\mathbb{\Psi}_W{}^0: \mathbb{Y}|W \longrightarrow \mathbb{X}|W \times \mathbb{C}^p$ of $\mathbb{\Phi}|W$. The homogeneous polynomial relations among $g_0, \ldots, g_p$ (over $\mathcal{O}_X(U)$) are also satisfied by $h_0, \ldots, h_q$ so that $\mathbb{\Psi}_W{}^0$ induces a lifting $\mathbb{\Psi}_W: \mathbb{Y}|W \longrightarrow B_I(\mathbb{X})$ of $\mathbb{\Phi}|W$. We can glue such local liftings to a global $\mathbb{\Psi}: \mathbb{Y} \longrightarrow B_I(\mathbb{X})$ in a standard way.

Any other local lifting $\mathbb{\Theta}_W$ of $\mathbb{\Phi}$ determines a ratio $\theta_W(T_0): \cdots : \theta_W(T_p)$. This ratio satisfies the relations $\phi(g_i)\theta_W(T_j) = \phi(g_j)\theta_W(T_i)$ which originate from $G_iT_j = G_jT_i$. Since $\phi(g_0)$ is a nonzerodivisor, the ratio $\phi(g_0): \cdots : \phi(g_p)$ is definite. Hence $\theta_W(T_0): \cdots : \theta_W(T_p) = \mathbb{\Psi}_W(T_0): \cdots : \mathbb{\Psi}_W(T_p)$ and $\mathbb{\Psi}$ and $\mathbb{\Theta}$ coincide. ∎

Now, as an extension of Hironaka's definition of the category of coherent ideal sheaves on CSs (see [H1], 0, §2), let us define *the category* $\mathbb{FCSI}^c$ *of coherent ideal sheaves on* FCSs. An objects of $\mathbb{FCSI}^c$ is a pair $(\mathbb{X}, I)$ of an FCS $\mathbb{X}$ and an coherent ideal sheaf I on $\mathbb{X}$. A morphism $\Phi^\sim \in \mathbb{FCSI}^c : (\mathbb{X}, I) \longrightarrow (\mathbb{Y}, J)$ is a morphism $\Phi \in \mathbb{FCS}$ with $\Phi^{-1}J = I$. Let $\mathbb{FCSI}^I$ denote *the category of invertible sheaves on* FCSs, which is the full subcategory of $\mathbb{FCSI}^c$.

3.3. Corollary. Let $\Phi^\sim : (\mathbb{X}, I) \longrightarrow (\mathbb{Y}, J)$ be a morphism in $\mathbb{FCSI}^c$ and $\Pi : \mathbb{X}' \longrightarrow \mathbb{X}$ and $\Theta : \mathbb{Y}' \longrightarrow \mathbb{Y}$ blowings up of $\mathbb{X}$ and $\mathbb{Y}$ with centers I and J respectively. Then we have the following.

(i) There exists a unique lifting $\Phi' : \mathbb{X}' \longrightarrow \mathbb{Y}'$ of Φ and Φ' defines a morphism $\Phi'^\sim : (\mathbb{X}', \Pi^{-1} I) \longrightarrow (\mathbb{Y}', \Theta^{-1} J)$ in $\mathbb{FCSI}^I$.

$$\begin{array}{ccc} (\mathbb{X}', \Pi^{-1} I) & \xrightarrow{\Phi'^\sim} & (\mathbb{Y}', \Theta^{-1} J) \\ \Pi^\sim \downarrow & & \downarrow \Theta^\sim \\ (\mathbb{X}, I) & \xrightarrow[\Phi]{\sim} & (\mathbb{Y}, J) \end{array}$$

(ii) The correspondence $\Phi^\sim \longrightarrow \Phi'^\sim$ defines an idempotent covariant functor from $\mathbb{FCSI}^c$ into $\mathbb{FCSI}^c$ and the image is $\mathbb{FCSI}^I \subset \mathbb{FCSI}^c$.

(iii) If Φ is a closed embedding, Φ' is also so.

If $\mathbb{X}$ is a subspace of $\mathbb{Y}$ in (iii), $\mathbb{X}'$ is a subspace of $\mathbb{Y}'$. We call $\mathbb{X}'$ the *strict transform* of $\mathbb{X}$. If the center I is nilpotent, the strict transform is vacuous.

Proof. (i) The first assertion follows from (3.2) and the fact that $I' \equiv \mathbb{H}^{-1} I = \mathbb{H}^{-1} \Phi^{-1} J$ is invertible. The second from

$$\Phi'^{-1} J' = \Phi'^{-1} \Theta^{-1} J = \mathbb{H}^{-1} \Phi^{-1} J = I' \quad (J' \equiv \Theta^{-1} J).$$

(ii) Functoriality follows from the universality in (3.2). It is idempotent by (3.1), (iii), (iv).

(iii) If Φ is a closed embedding, we may consider $\mathbb{X}$ as a closed subspace of $\mathbb{Y}$. Let L be the ideal sheaf defining $\mathbb{X} \subset \mathbb{Y}$. Since the canonical homomorphism $(J+L)/L \longrightarrow I$ is an isomorphism, there is a surjective ϕ-homomorphism from G_J onto G_I. This induces an embedding from $\mathbb{X}' \equiv \mathrm{B}_I(\mathbb{X})$ into $\mathbb{Y}' \equiv \mathrm{B}_J(\mathbb{Y})$. ∎

If $\mathbb{X}$ is reduced, it has the normalization $\mathbb{X}^{\sim}$ ([B_2], §3) and we can define the (global) irreducible components of $\mathbb{X}$ as the images of the connected components of $\mathbb{X}^{\sim}$ with the reduced structure. If U is a sufficiently small neighborhood of ξ, the irreducible components of $\mathbb{X}|U$ (or germs of them at ξ) correspond to the minimal primes of $O_{\mathbb{X},\xi}$ bijectively. We call a reduced and irreducible FCS *integral*.

<u>3.4.</u> <u>Proposition.</u> Let $\mathbb{H} \in \mathbb{FCS}:\mathbb{X}' \longrightarrow \mathbb{X}$ be a blowing up with center I.

(i) If $\mathbb{X}$ is reduced, then $\mathbb{X}'$ is also so and $\mathbb{X}'_\eta$ is the germ of the union of the strict transforms of the irreducible components of $\mathbb{X}_\xi$ ($\xi \equiv |\mathbb{H}|(\eta)$).

(ii) The canonically induced morphism $\mathbb{H}_{red} \in \mathbb{FCS}:\mathbb{X}'_{red} \longrightarrow \mathbb{X}_{red}$

is isomorphic to the blowing up whose center is the inverse image ideal sheaf of I.

Proof. (i) If $\mathbb{X}'$ is not reduced on $|\mathbb{H}|^{-1}(U)$ and if U is small, there exists a homogeneous polynomial $F(T_0,\dots,T_p)\in\sqrt{K(U)}\setminus K(U)$. By the definition of $K(U)$, $F(g_0,\dots,g_p)\neq 0$ $(g_i\equiv \iota_U(T_i))$ belongs to the nilradical. This contradicts the assumption that $\mathbb{X}$ is reduced and proves that $\mathbb{X}'$ is reduced. If $\mathbb{X}'$ is not the union, there exists a formal curve $\mathbb{K}'\subset\mathbb{X}'$ which is not contained in the union and the exceptional space by (1.2). Its image $\mathbb{K}$ is included in an irreducible component of $\mathbb{X}$ and not in the center. Then $\mathbb{K}'$ must be included in the strict transform of the component by the uniqueness of lifting (3.2), a contradiction.
(ii) This follows from the universalities of blowing up and reduction. ∎

4. Dimension of a germ of an FCS

We define the *dimension* of an FCS $\mathbb{X}$ at ξ (or *dimension* of the germ $\mathbb{X}_\xi$) by the Krull dimension of the local ring: $\dim \mathbb{X}_\xi \equiv \dim \mathcal{O}_{\mathbb{X},\xi}$.

Suppose that A is an integral domain. We define the *rank* of an A-module M as the maximal number r such that there exists an injective A-homomorphism $A^r \longrightarrow M$. If $\bar{A}$ denote the fields of fractions the rank is equal to the dimension of the $\bar{A}$-vector space $\bar{A}\otimes_A M$. This rank is denoted by $\mathrm{rank}_A M$.

Let A be a formal analytic algebra and $\Omega(A)$ the space of *Pfaffian forms* on A (=the universally finite differential module). This can be defined in the same way as in the case of analytic algebras and has similar properties ([B_2], (1.8)).

Nagata has proved that integral analytic algebras are formally integral i.e. its maximal-ideal-adic completion is also integral. This can be generalized to formal analytic algebras by (iii) of the following.

4.1. Theorem. Let A be a formal analytic algebra A and $\hat{A}$ its completion with respect to the maximal ideal. Then we have the following.

(i) A is excellent and Henselian;

(ii) A has the approximation property for algebraic equations (cf. [P], [Ro] for this property);

(iii) If p is a prime ideal, $\hat{p}\equiv p\hat{A}$ is also so.

Proof. (i) The first assertion was proved by Bingener $[B_2]$, (1.10). (Using the fact that excellence is equivalent to the existence of the universally finite differential module (cf. [SS], (8.10).) Suppose that A is the completion of an analytic algebra B with respect to the I-adic topology (I: an ideal of B). Since analytic algebras are known to be Henselian ([N], (45.6)), $A/IA \cong B/I$ is also so. Then A is Henselian by [R], p.5, Exercise. General formal analytic algebras are homomorphic images of completions of analytic algebras. It is easy to see that homomorphic images of Henselian rings are again Henselian.

(ii) It is known that a Henselian excellent ring has the approximation property (see Rotthaus [Ro], (4.2), Popescu [P]).

(iii) $\hat{p}$ is not prime

$\Longrightarrow \exists f, \exists g \in (A/p)^{\wedge}$ s.t. $f=0$, $g=0$ and $fg \neq 0$

$\Longrightarrow \exists f, \exists g \in A/p$ s.t. $f=0$, $g=0$ and $fg \neq 0$

$\Longrightarrow$ p is not prime

(We have applied (ii) to the algebraic equation $XY=0$ to show the third arrow. We could have proven (iii) directly from (i) in EGA style, cf. [R], XI, Proof of Th.3) ∎

<u>4.2.</u> <u>Theorem.</u> (cf. [SS], (4.1)) If A is an integral formal analytic algebra, $\mathrm{rank}_A \Omega(A) = \dim A$.

Proof. Let $\hat{A}$ denote the full completion of A. It is known that $\Omega(\hat{A}) \cong \hat{A} \otimes_A \Omega(A)$ ([SS], (1.6)). Since $\hat{A}$ is A-flat and integral by (4.1), $\mathrm{rank}_A \Omega(A) = \mathrm{rank}_{\hat{A}} \Omega(\hat{A})$. Completion preserves Krull dimension also (cf. [N], (17.12)). Therefore we may assume that A is fully formal i.e. A is a residue class algebra of $\mathbb{C}[\![x_1, \ldots, x_n]\!]$. Then it is well-known that, if $r = \dim A$, there exists a finite monomorphism $B \equiv \mathbb{C}[\![x_1, \ldots, x_r]\!] \longrightarrow A$ with (an application of the Weierstraß preparation theorem). Since $x_1, \ldots, x_r$ form a system of parameters of A, $dx_1, \ldots, dx_r$ are linearly independent over A ([SS], (8.12)). Any element $f \in A$ satisfies a monic polynomial relation over B. Taking the total derivative of this relation, we see that df is linearly dependent on $dx_1, \ldots, dx_r$ over A. Hence $\mathrm{rank}_A \Omega(A) = r = \dim A$. ∎

<u>4.3.</u> <u>**Theorem.**</u> If $\mathbb{X} \in \mathbb{FCS}$ is integral, $\dim \mathbb{X}_\xi$ is constant on $|\mathbb{X}|$. For general $\mathbb{X} \in \mathbb{FCS}$, $\dim \mathbb{X}_\xi$ is upper semicontinuous with respect to ξ.

Proof. The second assertion easily follows from the first. Since normalization preserves dimensions of local irreducible components, we may assume that $\mathbb{X}$ is normal (cf. [B_2], §3 for normalization, [Mat], Exercise 9.2 for invariance of dimension). Then $\mathbb{X}_\xi$ is integral at all $\xi \in |\mathbb{X}|$. $\Omega(\mathcal{O}_{\mathbb{X},\xi})$ ($\xi \in |\mathbb{X}|$) form a coherent sheaf $\Omega(\mathcal{O}_{\mathbb{X}})$ ([B_2], §1). Hence there exists a local exact sequence:

$$\mathcal{O}_{\mathbb{X}}(U)^p \xrightarrow{\lambda} \mathcal{O}_{\mathbb{X}}(U)^q \longrightarrow \Omega(\mathcal{O}_{\mathbb{X}})(U) \longrightarrow 0$$

Let $F_n \subset \mathcal{O}_{\mathbb{X}}(U)$ denote the ideal sheaf (*Fitting ideal sheaf*) generated by the $(q-n)\times(q-n)$ minorants of the matrix representing the left arrow. (This does not depend upon the choice of the exact sequence.) Let $\mathbb{Y}_r \subset \mathbb{X}$ denote the subspace defined by F_r $(\phi \subset \mathbb{Y}_{q-1} \subset \mathbb{Y}_{q-2} \subset \ldots \subset \mathbb{Y}_0 \subset \mathbb{X})$. Obviously

$$\operatorname{rank} \Omega(\mathcal{O}_{\mathbb{X},\xi}) \geqq r+1 \Leftrightarrow \operatorname{rank} \lambda \leqq q-r-1$$

$$\Leftrightarrow (F_r)_\xi = 0 \Leftrightarrow (\mathbb{Y}_r)_\xi = \mathbb{X}_\xi .$$

Since $\mathbb{X}$ is integral, if we put

$$s \equiv \max\{\operatorname{rank} \Omega(\mathcal{O}_{\mathbb{X},\xi}) : \xi \in |\mathbb{X}|\}-1,$$

we have $|\mathbb{X}| = |\mathbb{Y}_r|$ $(r \leqq s)$. Then $\dim \mathbb{X}_\xi = \operatorname{rank} \Omega(\mathcal{O}_{\mathbb{X},\xi}) = s+1$ everywhere by (4.2). ▮

5. Dimensions of images of blowings up

Let $\phi : B \longrightarrow A$ be a homomorphism between integral formal analytic algebras. By the universality of $\Omega(B)$, ϕ naturally induces a homomorphism $\phi^1 : \Omega(B) \longrightarrow \Omega(A)$ compatible with ϕ (cf. [SS]). We define the *generic rank* of ϕ by $\operatorname{grk} \phi \equiv \operatorname{rank}_A A\phi^1(\Omega(B))$. Obviously $\operatorname{grk} \phi \leqq \min\{\dim A, \dim B\}$. If $\Phi : \mathbb{X} \longrightarrow \mathbb{Y}$ is a morphism between FCSs and if $\xi \in |\mathbb{X}|$, the *generic rank* $\operatorname{grk} \Phi_\xi$ of Φ at ξ is, by definition, equal to that of the induced homomorphism $\phi_\xi : \mathcal{O}_{\mathbb{Y},\eta} \longrightarrow \mathcal{O}_{\mathbb{X},\xi}$ $(\eta \equiv |\Phi|(\xi))$. This is a generalization of Gabrielov's generic rank in analytic geometry (cf. [G]).

5.1. Lemma. Let A be an integral formal analytic algebra and $H_1, \ldots, H_n$ ($n=\dim A$) a system of parameters of A. If G is a nonzero element of A, $d(G^q H_1), \ldots, d(G^q H_n)$ form a basis for the the $\bar{A}$-vector space $\bar{A}\otimes_{\bar{A}}\Omega(A)$ for any $q\in\mathbb{Z}$ possibly except one value.

Proof. Since A is an integral domain, $dH_1, \ldots, dH_n$ generate a submodule with the full rank of $\Omega(A)$ ([SS], (8.12)). Hence there exists an expression $1\otimes dG = G_1\otimes dH_1 + \cdots + G_n\otimes dH_n$ in $\bar{A}\otimes_A\Omega(A)$ and we have

$$1\otimes d(G^q H_1) = qG^{q-1}H_1(G_1\otimes dH_1 + \cdots + G_n\otimes dH_n) + G^q\otimes dH_1.$$

By a calculation, we see that these are independent unless $nG + q(G_1H_1 + \cdots + G_nH_n) = 0$. ∎

5.2. Theorem. Let $\mathbb{II}\in\mathbb{FCS}:\mathbb{X}'\longrightarrow\mathbb{X}$ be a blowing up of an FCS and $\mathbb{Y}_\eta$ be an irreducible component of $\mathbb{X}'_\eta$. Then we have the following:

(i) If $|\mathbb{II}|(\eta)=\xi$, $\dim\mathbb{Y}_\eta = \operatorname{grk}\mathbb{II}|\mathbb{Y}_\eta \leqq \dim\mathbb{X}_\xi$.

(ii) If $\mathbb{X}_\xi$ is equidimensional, then $\dim\mathbb{Y}_\eta = \operatorname{grk}\mathbb{II}|\mathbb{Y}_\eta = \dim\mathbb{X}_\xi$ and $\mathbb{X}'_\eta$ is equidimensional as well as $\mathbb{X}_\xi$.

Proof. (i) Let U be a small neighborhood of ξ such that there exists a degree-preserving $\mathcal{O}_{\mathbb{X}}(U)$-epimorphism

$$\iota(\mathrm{U}) : \mathcal{O}_{\mathbb{X}}(\mathrm{U})[\mathrm{T}_0, \ldots, \mathrm{T}_p] \longrightarrow G_I(\mathrm{U}) \equiv \bigoplus_{k \geq 0} I(\mathrm{U})^k$$

with finitely generated kernel $K(\mathrm{U})$ and $G_I(\mathrm{U})_\xi = \bigoplus_{k \geq 0} I_\xi{}^k$. $\mathbb{X}' | \mathrm{V}$ ($\mathrm{V} \equiv |\mathbb{H}|^{-1}(\mathrm{U})$) is the formal complex space associated to the algebraic scheme $\mathbb{Z} \equiv \mathrm{Proj}\ G_I(\mathrm{U})$. We put

$$g_i \equiv \iota_{(\mathrm{U})}(\mathrm{T}_i) \in I(\mathrm{U}), \qquad \mathrm{G}_i \equiv \pi(g_i) \in \mathcal{O}_{\mathbb{X}'}(\mathrm{U}').$$

Let $\mathrm{H}_1, \ldots, \mathrm{H}_d$ ($d = \dim \mathbb{X}'_\eta$) be a system of parameters of the algebraic local ring $\mathrm{A}_{\mathbb{Z}, \eta}$. We may assume that $\mathrm{T}_0 \neq 0$ at η. Then H_i can be expressed as $\iota_{(\mathrm{U})}(\sum_{|\alpha| = q} h^i{}_\alpha (\mathrm{T}^\alpha / \mathrm{T}_0{}^q))$ ($\mathrm{T} = (\mathrm{T}_0 \ldots \mathrm{T}_p)$, $h^i{}_\alpha \in \mathcal{O}_{\mathbb{X}}(\mathrm{U})$), where $\iota_{(\mathrm{U})}$ is naturally extended to $\mathcal{O}_{\mathbb{X}}(\mathrm{U})[\mathrm{T}_0, \ldots, \mathrm{T}_p, 1/\mathrm{T}_0]$. G_0 and H_i can be considered as elements of the analytic local ring $\mathrm{O}_{\mathbb{Y}, \eta}$. $\mathrm{H}_1, \ldots, \mathrm{H}_d$ include a system parameters of $\mathcal{O}_{\mathbb{Y}, \eta}$. Since $\iota_{(\mathrm{U})}(g_0(\mathrm{T}_j/\mathrm{T}_0)) = g_j$, $\mathrm{G}_0{}^q \mathrm{H}_i$ belong to $\pi(\mathcal{O}_{\mathbb{X}, \xi})$ and

$$d(\mathrm{G}_0{}^q \mathrm{H}_i) \in d(\pi(\mathcal{O}_{\mathbb{X}, \xi})) = \pi^1(d\mathcal{O}_{\mathbb{X}, \xi})$$

Since G_0 is a nonzerodivisor even in $\mathcal{O}_{\mathbb{Y}, \eta}$, $d(\mathrm{G}_0{}^q \mathrm{H}_1), \ldots, d(\mathrm{G}_0{}^q \mathrm{H}_d) \subset \pi^1(\Omega(\mathcal{O}_{\mathbb{X}, \xi}))$ generate $\bar{\mathcal{O}}_{\mathbb{Y}, \eta} \otimes \Omega(\mathcal{O}_{\mathbb{Y}, \eta})$, except one bad value of q by (5.1). Thus we have $\dim \mathbb{Y}_\eta = \mathrm{grk}\ \mathbb{H} | \mathbb{Y}_\eta \leq \dim \mathbb{X}_\xi$.

(ii) By (3.4) we may assume that $\mathbb{X}$ is reduced and I_ξ is not included in a minimal prime ideal of $\mathcal{O}_{\mathbb{X}, \xi}$. (Otherwise, the blowing up of the corresponding component is vacuous.) The completion of a local ring preserves the dimension (cf. [N], (17.12)) and the decomposition into the irreducible components by (4.1). Hence, if $\mathbb{X}_\xi$ (or $\mathcal{O}_{\mathbb{X}, \xi}$) is equidimensional, it is formally so (=quasi-unmixed i.e. the completion is equidimensional). All the properties of the germ of $\mathbb{X}'$ along $\mathbb{E}$ is reflected by those of the germ $C_I(\mathbb{X})_F$ of the cone $C_I(\mathbb{X})$ along $F \equiv |\Pi^0|^{-1}(\xi)$. All the properties of $C_I(\mathbb{X})_F$ are reflected by those of the graded ring $A \equiv \bigoplus_{k \geq 0} I_\xi{}^k$ or by Proj A. (These are intuitive observations. See (5.3) for a correct argument.) Since $\mathcal{O}_{\mathbb{X},\xi}$ is equidimensional and since I_ξ is not included in a minimal prime ideal, A is also formally, and hence plainly, equidimensional of dimension $r+1$ ($r \equiv \dim \mathbb{X}_\xi$) ([HIO], (18.23), (9.7)). Let $p \subset A$ be hogeneous prime ideal of coheight 1 corresponding to η and $q \subset A$ the minimal homogeneous prime ideal corresponding to $\mathbb{Y}_\eta$. Of course, $q \subset p$. Since a formally equidimensional local ring is universally catenary (cf. [Mat], (31.6), [HIO], (18.17)), the coheight of q is $r+1$ and there exists a chain of prime ideals of length r (maximal) which connects q and p (consider the localization A_p). Then there exists a chain of *homogeneous* prime ideals in A/q which connects 0 and p/q (cf. [Mat], (13.7)). Its inverse image in A form a chain of homogeneous prime ideals of A which connects q and p. This proves that $\dim \mathbb{Y}_\eta \geqq r$, first algebraically and then formal-analytically. Then $\mathbb{X}'_\xi$ is obviously equidimensional of dimension r. ∎

5.3. Remark. Let X be a ringed space, $\mathbb{COH}(X)$ the category of sheaves of $\mathcal{O}_X$-module of finite presentation whose morphisms are defined to be $\mathcal{O}_X$-homomorphisms and $\mathbb{COHI}(X)$ the category of ideal sheaves of finite presentation on X whose morphisms are inclusions. $\mathbb{COHI}(X)$ can be seen as the *category of subobjects* of $\mathcal{O}_X \in \mathbb{COH}(X)$. (Then an object of $\mathbb{COHI}(X)$ is a monomorphism $\theta \in \mathbb{COH}(X)$: $\mathcal{F} \longrightarrow \mathcal{O}_X$. A morphism of $\mathbb{COHI}(X)$ comes from a monomorphism in $\mathbb{COH}(X)$. By the terms of [M], p. 122, $\mathbb{COHI}(X)$ is the category of equivalence classes of monics with codomain $\mathcal{O}_X$.)

Let $\mathbb{X}$ be a Stein FCS and $\mathbb{Z}$ an A-scheme ($A \equiv \mathcal{O}_{\mathbb{X}}(|\mathbb{X}|)$) locally of finite presentation. Bingener ([B_2], (4.5): The existence theorem) has proved the following: If $\mathbb{Z}$ is proper over Spec A and if $C \subset |\mathbb{X}|$ is a semianalytic Stein compact subset, there exists a canonical equivalence

$$E: \mathbb{COH}(S) \longrightarrow \varinjlim \mathbb{COH}(\mathbb{Z}^{an}|U) \qquad (S \equiv \mathbb{Z} \times_{\mathrm{Spec}\ A} \mathrm{Spec}\ \mathcal{O}_{\mathbb{X}}(C)),$$

of categories, where the limit is taken over the directed system of all Stein neighborhoods U of C. Monomorphisms are characterized by left cancellability: a property defined by the terms of the categories. Then $f \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ is a monomorphism if and only if $E(\mathcal{F})$ is so. Therefore E induces an isomorphism

$$E': \mathbb{COHI}(S) \longrightarrow \varinjlim \mathbb{COHI}(\mathbb{Z}^{an}|U).$$

In the situation of (5.2), this implies that the lattis structures of (a) the family of subspaces of Proj $\bigoplus I_\xi^k$ and (b) the family of germs of subspaces of Proj $G_I(\mathbb{X}|U)$ around F are isomorphic, where U runs over the directed system of all Stein neighborhoods of ξ.

5.4. Remark. Consider a part of blowing up $\Pi : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$, $x' = x$, $y' = xy$, $z' = xz$. Let $t(y)$ be a transcendental holomorphic function. Osgood has pointed out that x, xy, $xyt(y)$ have no formal relation at 0 (cf. [GR_1]). Hence no germ of a thin subspace of $\mathbb{C}^3$ at 0 includes the image of the germ of the thin subspace $\{z = yt(y)\}$ at 0. Thus we can not expect that a thin subspace at a point is not always mapped into a thin subspace by a blowing up. But we can expect so if the subspace is defined in a neighborhood of the entire $|\mathbb{E}|$ as follows.

5.5. Proposition. Let $\mathbb{\Pi} \in \mathbb{FCS} : \mathbb{X}' \longrightarrow \mathbb{X}$ be a blowing up and $\mathbb{Y}$ a thin subvariety defined in a neighborhood of the exceptional space $\mathbb{E}$. Then $\mathbb{\Pi}\,|\,\mathbb{Y}$ factors through an everywhere thin subspace $\mathbb{Z}$ of an open neighborhood of the center.

Proof. Let $\mathbb{Y}^* \subset C_I(\mathbb{X}|U)$ be the cone subspace corresponding to $\mathbb{Y}$. Since $\mathbb{Y}$ is thin, there exists a homogeneous element $F(T_0, \ldots, T_p) \in \mathcal{O}_{\mathbb{X}}(U)[T_0, \ldots, T_p]$ such that $\iota\,(U)(F(T_0, \ldots, T_p))$ is active in $\mathcal{G}_I(U)$ and vanishes on $\mathbb{Y}^*$. Then $F(g_0, \ldots, g_p) \in \mathcal{O}_{\mathbb{X}}(U)$ is also active by the definition of $\mathcal{K}(U)$. We have only to define $\mathbb{Z}$ by $F(g_0, \ldots, g_p)$. ∎

6. Blowing up with an analytic center

The following is a part of a result of Ancona-Tomassini, which was proved in a sophisticated way.

6.1. Proposition. ([AT], (III.6)) Let $I \subset \mathcal{O}_X$ be a coherent ideal sheaf on $X \in \mathbb{CS}$ and S its analytic subset such that the subspace of X corresponding to I is contained in S. Let $\Theta \in \mathbb{FCS}:\mathbb{Y} \longrightarrow \hat{\mathbb{X}}_{|S}$ be the blowing up with center $I\hat{\mathcal{O}}_{X|S}$ and $\hat{\Pi} \equiv \hat{\Pi}_{T:S} : \hat{\mathbb{X}}'_{|T} \longrightarrow \hat{\mathbb{X}}_{|S}$ the completion of the blowing up $\Pi \in \mathbb{CS}: X' \longrightarrow X$ with center I, where $T = |\Pi|^{-1}(S)$. Then $\mathbb{Y}$ and $\hat{\mathbb{X}}'_{|T}$ are canonically isomorphic and $\hat{\Pi}$ coincides with Θ by this identification.

Proof. Since $I\hat{\mathcal{O}}_{X'|T}$ is invertible there exists a lifting $\Sigma : \hat{\mathbb{X}}'_{|T} \longrightarrow \mathbb{Y}$ of $\hat{\Pi}$ by the universality of Θ. By (3.3), the completion of the identity $\Phi \equiv \hat{1}_{S:|X|} \in \mathbb{FCS}:\hat{\mathbb{X}}_{|S} \longrightarrow X$ has a unique lifting $\Phi' \in \mathbb{FCS}:\mathbb{Y} \longrightarrow X'$. Since $|\mathbb{Y}|$ is mapped into T by $|\Phi'|$, we have the completion $\Psi \equiv \hat{\Phi}'_{|\mathbb{Y}|:T} : \mathbb{Y} \longrightarrow \hat{\mathbb{X}}'_{|T}$ with $\Phi' = \hat{1}_{T:|X'|} \circ \Psi$. Thus we obtain the following diagram.

$$\begin{tikzcd}
\hat{\mathbb{X}}'_{|T} \arrow[rr, "\hat{1}_{T:|X'|}"] \arrow[dd, "\hat{\Pi}"'] \arrow[dr, shift left, "\Sigma"] & & X' \arrow[dd, "\Pi"] \\
& \mathbb{Y} \arrow[ul, shift left, "\Psi"] \arrow[ur, "\Phi'"'] \arrow[dl, "\Theta"] & \\
\hat{\mathbb{X}}_{|S} \arrow[rr, "\Phi"'] & & X
\end{tikzcd}$$

Since

$$(\Sigma \cdot \Psi)^{-1}(\Phi \cdot \Theta)^{-1} I = \Psi^{-1} \cdot \hat{\Pi}^{-1} \cdot \Phi^{-1} I$$

$$= \Psi^{-1} \cdot \hat{1}_{T;|X'|}{}^{-1} \cdot \Pi^{-1} I = \Phi'^{-1} \cdot \Pi^{-1} I = (\Phi \cdot \Theta)^{-1} I,$$

we have $\Sigma \cdot \Psi = I_{\mathbb{Y}}$ by the universality of Θ. Then $\psi_{\zeta} \cdot \sigma_{\eta}$ $(\eta \in |\mathbb{Y}|,\ \zeta \equiv \Psi_{\eta}(\eta))$ is the identity. Similarly, by the universality of $\Pi \in \mathbb{CSC}\ \mathbb{FCS}$, we can prove $\Phi' \cdot \Sigma = \hat{1}_{T;|X'|}$. Then $\Psi \cdot \Sigma$ commutes with $\hat{1}_{T;|X'|}$. This implies $\sigma_{\eta} \cdot \psi_{\zeta} : \hat{\mathcal{O}}_{X'|T,\zeta} \longrightarrow \hat{\mathcal{O}}_{X'|T,\zeta}$ commutes with the canonical monomorphism $\hat{\iota}_{T;|X'|} : \mathcal{O}_{X',\zeta} \longrightarrow \hat{\mathcal{O}}_{X'|T,\zeta}$. Hence $\sigma_{\eta} \cdot \psi_{\zeta}$ is the identity on the image of $\hat{\iota}_{T;|X'|}$. This image is dense in $\hat{\mathcal{O}}_{X'|T,\zeta}$ with respect to the maximal-ideal-adic topology. Then we see that $\sigma_{\eta} \cdot \psi_{\zeta}$ is also the identity. Thus $\mathbb{Y}$ and $\hat{X}'_{|T}$ are canonically isomorphic. ∎

<u>6.2.</u> <u>Remark.</u> Note that this proposition implies Θ is the completion of an analytic morphism $Z \longrightarrow X$. If the center is not contained in the core, there is no canonical way to take Z.

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